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ELEMENTS IN SET THEORY

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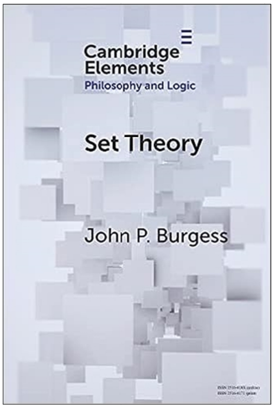

Another volume of the *Elements in Philosophy and Logic* series (edited by Bradley Armour-Garb and Frederick Kroon), published by Cambridge University Press, has been released. The author of *Set Theory* is John P. Burgess, affiliated as Professor of Philosophy at Princeton University. He specializes in logic and philosophy of mathematics. The *Elements* publishing series was created to “provide a dynamic reference resource for graduate students, researchers, and practitioners.” Thus, it is the publisher’s intention that successive volumes be used by both students and professional researchers. In addition, the publisher promotes the entire series as combining the best features of books and scholarly articles, providing a quick and concise study of a given topic. It can be expected that *Set Theory* will be useful for students and professional researchers, so it will provide both basic knowledge and the results of the latest research in set theory.

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The reviewed book was published in 2022 and is available online (in PDF format), as well as traditionally, as a paperback. The publisher allows downloading the publication as part of Cambridge Core subscription, so there is a good chance that a university library can get access to the digital version. The book contains sixty-eight pages of text, is divided into eleven short chapters, and in addition we get five reference pages of references. There is no introduction, the reader receives only a short abstract at the beginning. Also missing is a conclusion, a subject index and a personal index. From the brief acknowledgements on the last page, we learn that the author wrote this book with more than fifty years of experience in teaching and researching set theory.

The reviewed publication can be divided into two main parts, which are not highlighted in the table of contents, but a careful reading clearly indicates them. Chapters one to seven inclusive deal with issues that are basic and generally present in most textbooks on set theory. Chapters eight through eleven, on the other hand, deal with advanced issues that are unlikely to be common knowledge even among philosophers and mathematicians.

The first part of the book begins with a chapter entitled "Historical roots," which is a basic discussion of the fundamental concepts of set theory: ordinals and cardinals, denumerable sets, the theory of relations and also the concept of Cantor's diagonal proof is recalled. In chapter 2, entitled "The Notion of Set," the common concept of a set as a certain collection and related problems are first discussed. Russell's paradox and the way to avoid it by means of type theory are mentioned, this solution is called layered hierarchy of sets. The second part of the chapter is devoted to presenting the cumulative concept of set, where the ZFC theory is mentioned. The basic logical apparatus, useful abbreviations and definitions are also introduced here and collected in several tables. The next chapter, "The Zermelo-Fraenkel Axioms," presents ZFC – the axiomatic theory of sets. More axioms of this theory are cited, along with a few sentences of commentary on each. In section 3.2, "Motivation," the author briefly recalls the inspiration that accompanied the founders of ZFC, as well as a discussion regarding the axioms and their relation to the concept of cumulative set. In this way, the reader is introduced to the subsequent ideas of set theory and some of the basic difficulties encountered by the theory. Chapter 4, "Immediate Consequences," can be regarded as a school script of the basic issues of set theory. The algebra of sets is discussed along with its relation to Boolean algebras, as well as the theory of relations. This section presents (usually in tables) the most useful concepts and symbols with definitions. Chapter 5, "Number Systems within Set Theory," discusses the reduction of numbers to the concepts of set theory. Following the section on "history of modern mathematics in about 300 words" (p. 22), the author presents how to bring the successive levels of mathematics, from complex numbers, through real and quantifiable numbers, to natural numbers. In this section, Peano's axiomatization of natural number arithmetic is briefly presented and discussed, along with some commentary on how natural numbers can be reduced to certain hierarchies of sets. The chapter "Infinities" discusses the cardinals, ordinal types of sets, ordinals and the aleph scale. Basic theorems (e.g., the Cantor–Bernstein Theorem), properties and possible operations in these areas are presented. Chapter 7, “The Axiom of Choice,” offers further theorems and properties of sets, which are presented in the context of a discussion of the axiom of choice. In addition to presenting various versions of this axiom and some of its paradoxical consequences, the Well-Ordering Principle and Zorn’s lemma are discussed along with proofs as well.

Chapter 8, “Topics in Higher Set Theory,” begins the second, more advanced part of the book. We first read an introduction that outlines the three main areas of research within higher set theory: (1) descriptive set theory, (2) continuum theory, and (3) combinatorial set theory. Each of these areas relates to results that cannot be achieved using the standard ZFC axioms. Extending the axioms to include theorems about large cardinals results in outcomes on the descriptive set theory field. The forcing axioms, on the other hand, refer to the area called continuum theory and to a lesser extent to combinatorial set theory. The section “The Descriptive Set Theory” deals mainly with basic topological concepts, including spaces, Polish space, open and closed sets, perfect sets and their main properties, as well as the Cantor–Bendixson Theorem. The subsection “Continuum Theory” describes, in particular, the problem of the continuum hypothesis and its independence from the ZFC axiomatics. The table lists several metatheoretical theorems relating to this topic. In the last section (“Combinatorial Set Theory”) of the chapter beginning with an introduction of the generalized continuum hypothesis, we get König’s Infinity Lemma, the Infinite Ramsey’s Theorem, Finite Ramsey’s Theorem along with sketches of proofs and some of their consequences. The next chapter, “Metamathematics of Set Theory,” first discusses issues related to models and the concept of truth for set theory. Next, inaccessible cardinals are presented, along with a discussion regarding their relation to ZFC. Mention is made here of Saunders Mac Lane’s category theory, as well as the Levy Reflection Principle. A sketch of the proof of the relative consistent of ZF with the axiom of choice, based on internal models, is also presented. The chapter concludes with some remarks on the status of the continuum hypothesis, which were achieved by Cohen’s forcing method. Martin’s axiom and some of its consequences are also recalled. Chapter 10, “Large Cardinals and Determinacy,” describes discoveries at the interface of descriptive set theory and large cardinal theory. It talks about Tarski’s weakly compact cardinals, Ulam’s measurable cardinals, and only Woodin cardinals and supercompact cardinals are signaled. Also described are some discoveries within the game theory field, which are connected with issues of set theory. This is mainly about the infinite games of perfect information coming from the Polish scientists, as well as Davis Theorem and Gale-Stewart Theorem, whose proof is only sketched. Finally, there is also a note on the latest research (2021) under the so-called Woodin axiom. In the final chapter, “Concluding Philosophical Remarks,” the author presents some philosophical questions that are posed regarding the open issues of set theory. Possible answers to these questions are outlined herein. The reader, on the other hand, does not receive any resolution or even Burgess’ personal response to these issues.

The reviewed publication is intended by the publisher to be a concise study of set theory, useful for both students and professional researchers. The first part of the book, chapters 1–6, undoubtedly offers an introduction to the basic issues of the set theory. Noteworthy is the way in which the material is presented, the historical references that appear, along with references to classical authors in the field. The tables and figures are also interesting, as they allow one to understand the material discussed and provide a good reference point from which to use in further reading and other scholarly work. There are also occasional original passages that are worth recommending in the first place. Among other things, Burgess, as part of his presentation of the theory of cardinals, recalls the Cantor–Bernstein theorem. It is definitely worth noting the sketch of the proof of this theorem, which is presented as a series of exercises. It is rather uncommon in other textbooks, and remarkable because it uses only the elementary concepts apparatus of set theory, related to the properties of relations and their compositions.

Some comments from the author are also worthy of note in *Set Theory*. For example, Burgess introduces the definition of an ordered pair, which is called here the Wiener–Kuratowski definition (p. 15), and then defines the relation as a set of given ordered pairs. Since there is no set of all ordered pairs such that "$x \in y$", then $\in$ is not a relation in the sense discussed here. The same is true of the relation of inclusion $\subset$ (cf. p. 19). Thus, the concept of a relation can be defined using the terms of ZFC, but $\in$ is not a relation of that type. A remark of this type is unlikely to be found in other set theory textbooks. In addition, we will make an equally interesting remark elsewhere regarding the commonly accepted statement that ZFC is the basis theory and terms apparatus for all mathematics, especially because natural numbers can be reduced to its terms. And as is known, as also shown in the reviewed book, to the arithmetic of natural numbers are reduced the other levels of classical mathematics. Nevertheless, Peano's axioms are sufficient to describe the structure of natural numbers. On the other hand, there is much more in ZFC than just a basis for natural numbers (cf. p. 28).

The second part of *Set Theory*, chapters 8 through 11, differs significantly from the first. On the one hand, we get an overview of the issues of high-level set theory, arranged in a certain predetermined sequence. This order is outlined clearly by reference to the underlying ZFC theory. This allows the reader to get a better idea of the problems of, for example, topology, or the issue of the continuum hypothesis and its derivatives. This way of presentation unquestionably has an advantage over some kind of encyclopedic compilation of these issues, or in some respects also over a comprehensive companion in which many authors comment on the latest discoveries in the field. After all, here we have a compact 68 pages, compared to, for example, the 2194 pages in *Handbook of Set Theory* (FOREMAN and KANAMORI 2010). There can be no doubt that after reading these chapters, the reader will have some idea of the advanced achievements of set theory. On the other hand, a presentation like this also has its weaknesses. The issues discussed in the second part of the book do not fit into a standard course in set theory, so they are not familiar even to professional philosophers or mathematicians. Many of the topics covered are merely mentioned in passing, without any deeper explanation, which is not the case for understandable reasons. It seems that the second part of the reviewed *Set Theory* is too advanced for students and those not mathematically trained. After all, one cannot expect the concept of Levy's hierarchy or Martin's Maximum to be understood without prior study. This is a clear drawback to this kind of presentation, but one presumes that the framework of this series proposed by the publisher is the reason.

Let us further note an important feature of the reviewed publication. Practically every chapter outlines proofs of the theorems presented. This applies both to the simplest basic theorems and to the theorems in the second part of the book, relating to high-level results of set theory. If the book is regarded as a handy resource for someone who has studied all these theorems and proofs in a course on set theory or reading source texts, it can prove very useful. If one wants to review a given theorem and recall the central idea of the proof of even an advanced theorem, Burgess's book *Set Theory* is the book to reach for. In this sense, this publication can be a useful item for professionals who need a handy set containing the essential theorems and guidelines on how to prove them.

Unfortunately, the reviewed volume of the *Elements* series also has some significant shortcomings. Numerous typographical errors appear. Annoyingly, one often comes across the erroneous indexing of consecutive symbols in formulas of the type $2^{\aleph_0}$, which are written this way: $2^{\aleph}{}_0$. This error is so common that the reader may get the impression that the editor has intended this notation. Similarly, the incorrect formal notation of the axioms of ZFC in unacceptable. The Axiom of Union (p. 10) and the Axiom of Foundation (p. 14) are written incorrectly. There are many other typographical errors of this kind that cannot be justified by the nature of this publication.

The references section is extensive and, for such a short work, takes up a whole five pages, but unfortunately not without errors. A careful reading shows that on page 63 the author refers the reader to some text on the Axiom of Determinacy by Mycielski and Świerczkowski from 1964, but this kind of position does not appear in the references. We are probably referring to the article "On the Lebesgue Measurability and the Axiom of Determinateness" (MYCIELSKI and ŚWIERCZKOWSKI 1964). There is another problem with the references. Recall that the *Elements* series, according to the publisher, is intended to provide insight into source material and references for those interested, yet on p. 41 we discover that from now on, in addition to indicating the names of authors, references to source texts will not be provided. The author gives two reasons for this decision: (1) reference texts are often not in English, (2) understanding them "often would require years of graduate study." This decision can hardly be justified. Commenting on this state of affairs, let us recall just one fact. In the first half of the twentieth century, logicians all over the world were learning the Polish language, and, for example, the University of Münster organized courses in Polish language (cf. BOCHEŃSKI 1994, 115) just to be familiar with the achievements of the Polish school of logic and mathematics, the results of which are also mentioned in the reviewed book. Is this standard no longer available today to those interested in set theory?

The overall assessment of the next volume published in the *Elements* series is therefore ambiguous. The book can be recommended to those who need handy notes on basic and advanced issues of set theory. Sketches of proofs and abbreviated key ideas of many topics are appreciated. The first seven chapters will be useful to those who have studied the basics of set theory, while the remaining chapters will only be helpful to professionals familiar with the subject. Finally, it should be said that Burgess undertook the very difficult task of capturing the achievements of the entire set theory in a few dozen pages – and he did that quite well. Most of the shortcomings pointed out in this review seem to lie with the publisher.

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